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White Dwarfs (Cont'd):

General Relativistic Corrections:

Our discussion of white dwarfs so far has been within the Newtonian gravity. At high densities, however, one should take effects from general relativity into consideration.

In the Newtonian gravity an equilibrium configuration can be achieved if the pressure is high enough. However, pressure is related to kinetic energy. According to Einstein theory of general relativity, all kinds of energy gravitate. Therefore, increasing pressure will make a contribution to the effective mass of the star, which will increase the gravitational force. This can lead to instability. Let us find when this becomes important.

Consider a polytrope with the equation of state $P = K\rho^{\gamma}$.

The total energy of the polytrope (in the Newtonian limit) is;

$$W = U + \Omega = k_0 M \frac{P_c}{\rho_c} - k_1 \frac{GM}{R^2} = k_2 M \rho_c^{\gamma-1} - k_3 G M^{\frac{5}{3}} \rho_c^{\frac{1}{3}}$$

Here the k_0, k_1, k_2, k_3 constants depend on the actual distribution of matter. The condition for equilibrium is:

$$\frac{\partial W}{\partial \rho_c} = 0 \Rightarrow \boxed{M \propto \rho_c^{\frac{3}{2}(\gamma - \frac{4}{3})}}$$

The condition for stability is:

$$\frac{\partial^2 W}{\partial \rho_c^2} > 0 \Rightarrow \boxed{\gamma > \frac{4}{3}}$$

Now we include the lowest order correction from general relativity. The internal energy (i.e. kinetic energy of the gas) is proportional to the gravitational potential energy in equilibrium (according to the Virial theorem). From mass-energy equivalence, it results in an effective mass for the star:

$$m_{eff} = \alpha_1 \frac{GM^2}{c^2 R}$$

The total energy of the star is now modified:

$$W = K_2 M \rho_c^{\gamma-1} - k_3 G M^{\frac{5}{3}} \rho_c^{\frac{1}{2}} - \alpha_2 \frac{G M m_{eff}}{R}$$

Writing $\rho_c = \alpha_3^3 \frac{M}{R^3}$, and defining $k_4 = \frac{\alpha_1 \alpha_2}{\alpha_3^2}$, we have:

$$W = k_2 M \rho_c^{\gamma-1} - k_3 G M^{\frac{5}{3}} \rho_c^{\frac{1}{2}} - k_4 \frac{G^2}{c^2} M^{\frac{7}{3}} \rho_c^{\frac{2}{3}}$$

Now the stability condition $\frac{\partial^2 W}{\partial \rho_c^2} > 0$ implies that:

$$\boxed{\gamma > \frac{4}{3} + \frac{2}{3}\left(\frac{k_4 \alpha_3}{k_3}\right) \frac{GM}{c^2 R}}$$

This shows that general relativity changes the critical value of γ to a larger value, hence increasing the domain of instability.

To determine the densities at which general relativistic corrections become relevant, we should find the value of γ for a relativistic degenerate electron gas. Starting from the definition of pressure,

$$P_e = \frac{8\pi}{3h^3} \int_0^{p_F} p \, v_{(p)} \, p^2 \, dp \quad , \quad v = c\left(1 + \frac{m_e^2 c^2}{p^2}\right)^{-\frac{1}{2}} \underset{(p \gg m_e c)}{\simeq} 1 - \frac{m_e^2 c^2}{2p^2}$$

We find:

$$P_e \simeq \frac{2\pi c}{3h^3} p_F^4 \left(1 - \frac{m_e^2 c^2}{p_F^2}\right) \Rightarrow \ln P_e \simeq 4 \ln p_F - \frac{m_e^2 c^2}{p_F^2} + \text{Const.}$$

Using the fact that $\rho \propto p_F^3$, we obtain:

$$\boxed{\gamma = \frac{d \ln P_e}{d \ln \rho} = \frac{4}{3} + \frac{2}{3}\left(\frac{m_e c}{p_F}\right)^2}$$

Omitting factors of unity, the white dwarf will be stable if:

$$\frac{GM}{c^2 R} < \left(\frac{m_e c}{p_F}\right)^2$$

Expressing p_F in terms of ρ, this condition becomes (we have omitted numerical factors of order unity):

$$\boxed{R > 10^8 \left(\frac{M}{M_\odot}\right)^{\frac{5}{9}} \nu_e^{-\frac{5}{9}} \text{ cm}}$$

Note that for a relativistic white dwarf at the Chandrasekhar mass limit (with $\nu_e = 2$, $M \simeq 1.4 M_\odot$) we have:

$$R \simeq 10^9 \rho_{rel}^{\frac{1}{3}} \rho^{-\frac{1}{3}} \text{ cm} \qquad (\rho_{rel} \simeq 10^6 \nu_e \text{ g cm}^{-3})$$

This implies that the general relativistic effects make the white dwarf unstable for $\rho > 1.2 \times 10^9 \text{ g cm}^{-3}$.

A more precise evaluation shows that the critical density is:

$$\boxed{\rho_c = 2.9 \times 10^{10} \left(\frac{\nu_e}{2}\right)^2 \text{ g cm}^{-3}}$$

For ^{56}Fe with $\nu_e = 2.154$, we have $\rho_c = 3.07 \times 10^{10} \text{ g cm}^{-3}$. The neutronization threshold for ^{56}Fe is $1.14 \times 10^9 \text{ g cm}^{-3}$. Therefore the instability due to relativistic corrections will be irrelevant since that from neutronization becomes important first. On the other hand, for ^{12}C with $\nu_e = 2$, we have $\rho_c = 2.56 \times 10^{10} \text{ g cm}^{-3}$. The neutronization threshold for ^{12}C is $\rho_c = 3.9 \times 10^{10} \text{ g cm}^{-3}$. Thus the general relativistic effects provide the limit on the stability in this case.

We conclude that the limiting configuration for white dwarfs is set by various physical processes that become important at high densities rather than the Chandrasekhar mass. The latter is of only theoretical significance and sets an absolute upper bound on the white dwarf mass.

Effects from Rotation and Magnetic Field:

So far, we have only considered spherically symmetric configurations. Thing can change in the presence of magnetic field or rotation. Here we discuss these effects.

Let us start with magnetic fields. The Virial theorem in the presence of magnetic fields becomes (see the third homework):

$$\Omega + 3M \left\langle \frac{P}{\rho} \right\rangle + \left\langle \frac{B^2}{8\pi} \right\rangle \frac{4}{3}\pi R^3 = 0$$

Here $\langle\,\rangle$ denotes averaging of a quantity over the star volume.

In the limit of high conductivity, the magnetic flux $\Phi_M \propto \langle B \rangle R^2$ is conserved $\left(\frac{\partial B}{\partial t} = 0 \text{ in this limit} \right)$. This allows us to write $\langle B^2 \rangle \propto \frac{\Phi_M^2}{R^4}$. The virial theorem in the non-relativistic ($P \propto \rho^{5/3}$) and relativistic ($P \propto \rho^{4/3}$) limits reduces to:

$$-\alpha_{3/2} \frac{GM^2}{R} + \beta_{3/2} \frac{M^{5/3}}{R^2} + \gamma_{3/2} \frac{\Phi_M^2}{R} = 0 \qquad \text{(non-relativistic)}$$

$$-\alpha_3 \frac{GM^2}{R} + \beta_3 \frac{M^{4/3}}{R} + \gamma_3 \frac{\Phi_M^2}{R} = 0 \qquad \text{(relativistic)}$$

Here the subscripts $\frac{3}{2}$, 3 refer to the polytropic index $n = \frac{1}{\gamma - 1}$ in the non-relativistic and relativistic limits respectively.

Comparing the first and third terms on the left-hand side of above equations, it is clear that the magnetic field effectively reduces G, and hence leads to expanding the star at any given mass. The effect is not very important in the non-relativistic limit. The more significant effect is in the relativistic limit. In this case, we have:

$$M^{\frac{2}{3}} = \frac{\beta_3}{\alpha_3 G R}\left(1 + \frac{\gamma_3 \Phi_M^2}{\beta_3 M^{\frac{4}{3}}}\right) \simeq \frac{\beta_3}{\alpha_3 G R}\left(1 + \frac{\gamma_3 \Phi_M^2}{\alpha_3 G M^2}\right)$$

We have assumed that $\mathcal{M} < |\Omega|$, where:

$$\mathcal{M} \equiv \left\langle \frac{B^2}{8\pi} \right\rangle \frac{4}{3}\pi R^3$$

For a polytrope with a polytropic index "n" we have:

$$\frac{\Delta R}{R} = \frac{3}{3-n}\,\frac{\Delta \mathcal{M}}{|\Omega|}$$

This may be integrated at constant n to yield:

$$\boxed{R = R_0 \exp\left(\frac{3}{3-n}\,\frac{\mathcal{M}}{|\Omega|}\right)}$$

Therefore the radius can increase significantly even with a small $\frac{\mathcal{M}}{|\Omega|}$ in the relativistic limit.

Next, we consider the effect of rotation. The rotational kinetic energy is given by $K_{rot} \propto \frac{J^2}{MR^2}$, where J is the (conserved) angular momentum. The virial theorem in the presence of rotation becomes:

$$-\alpha_{3/2}\frac{GM^2}{R}+K_{3/2}\frac{J^2}{MR^2}+\beta_{3/2}\frac{M^{\frac{5}{3}}}{R^2}=0 \qquad \text{(non-relativistic)}$$

$$-\alpha_3\frac{GM^2}{R}+K_3\frac{J^2}{MR^2}+\beta_3\frac{M^{\frac{4}{3}}}{R}=0 \qquad \text{(relativistic)}$$

In the non-relativistic limit rotation provides only a small correction to the mass-radius relationship. However, the change will be significant in the relativistic limit.

We see that in the absence of rotation, R drops out in the above equation, and hence the virial theorem can be satisfied for a specific value of M only. The rotational kinetic energy, however, scales with R differently. Therefore it is possible to satisfy the virial theorem for any M by decreasing R appropriately. (In reality, one has to be careful about various effects at high densities that can affect the stability, as we discussed, if R is very small.)

The virial theorem in the relativistic limit can be cast in the following form:

$$\alpha_3 \frac{GM^2}{R}\left(1-\frac{2K_{rot}}{|\Omega|}\right) = \beta_3 \frac{M^{\frac{4}{3}}}{R}$$

Solving for M, we find:

$$\boxed{M = \left[\frac{\beta_3}{\alpha_3 G\left(1-\frac{2K_{rot}}{|\Omega|}\right)}\right]^{3/2} = \frac{M_0}{\left(1-\frac{2K_{rot}}{|\Omega|}\right)^{3/2}}}$$

Here M_0 is the mass in the absence of rotation. The bound on M now rises from the maximum value allowed for the ratio $\frac{K_{rot}}{|\Omega|}$. The actual bound depends on the detaile assumptions about the shape of the configuration. In general, a range 0.14–0.26 is obtained for the maximum bound in different rotational configurations. Taking the ratio $\frac{K_{rot}}{|\Omega|} \approx 0.2$, the maximum mass is increased by a factor of ~ 2.

Thus rotation increases the maximum mass above the

the Chandrasekhar limit. This is expected as rotation has the tendency to oppose gravitational collapse. It can have significant effects for rapidly rotating white dwarfs.